Separable Approximation to Optical Potentials

Linda Hlophe

Advisor: Dr. Ch. Elster

Department of Physics and Astronomy
Ohio University
Athens, OH

Outline

- **Fourier transform of Wood-Saxon**
- **Momentum space: Comparison with FRESCO**
  - scattering phase shifts
  - bound states
- **Rank-1 separable potentials**
  - Unitary pole approximation (UPA)
  - Extension to positive energies (EST)

Wood-Saxon potential Fourier transform

- Coordinate space potential

$$U_{\rm nucl}(r) \;=\; V(r) \;+\; i\,(W(r) \;+\; W_s(r))$$

where

$$V(r) \;=\; \frac{-\,V_r}{1\;+\;\exp\left(\frac{r\,-\,R_r}{a_r}\right)};$$

$$W(r) \;=\; \frac{-\,V_i}{1\;+\;\exp\left(\frac{r\,-\,R_i}{a_i}\right)}$$

$$W_s(r) \;=\; \frac{-\,V_s\;\times\;4\;\times\;\exp\left(\frac{-\,(r\,-\,R_s)}{a_s}\right)}{\left(1\;+\;\exp\left(\frac{-\,(r\,-\,R_s)}{a_s}\right)\right)^2};$$

- From contour Integration (thanks R. C. Johnson)

$$\begin{aligned}\bar{V}(\mathbf{q}) \;&=\; \frac{V_r}{\pi^2}\Bigg\{\frac{\pi a e^{-\pi a q}}{q\left(1-e^{-2\pi a q}\right)^2}\left[R_0\left(1-e^{-2\pi a q}\right)\cos\left(qR_0\right)-\pi a\left(1+e^{-2\pi a q}\right)\sin\left(qR_0\right)\right]\\ &\quad -a^3 e^{-\frac{R_0}{a}}\left[\frac{1}{(1+a^2q^2)^2}-\frac{2e^{-\frac{R_0}{a}}}{(4+a^2q^2)^2}\right]\Bigg\}, \qquad (18)\end{aligned}$$

$$\begin{aligned}\bar{W}_s(\mathbf{q}) \;&=\; -4a_s\,\frac{V_s}{\pi^2}\left\{\frac{\pi a e^{-\pi a q}}{\left(1-e^{-2\pi a q}\right)^2}\right.\\ &\quad\left[\;\left(\pi a\left(1+e^{-2\pi a q}\right)-\frac{1}{q}\left(1-e^{-2\pi a q}\right)\right)\cos(qR_s)+R_s\left(1-e^{-2\pi a q}\right)\sin(qR_s)\right]\\ &\quad +\;\left. a^2 e^{-R_s/a}\left[\frac{1}{(1+a^2q^2)^2}-\frac{4e^{-R_s/a}}{(4+a^2q^2)^2}\right]\right\}, \qquad (19)\end{aligned}$$

Scattering properties

- Lippmann-Schwinger equation

$$t_l(k,k') = v_l(k,k') + \int_0^\infty dk''\, k''^2\, v_l(k,k'') \frac{1}{E - k'^2/2\mu + i\varepsilon}\, t_l(k'',k'),$$

- solved as linear system with zgesv from lapack (~50 pts)

- partial wave projection

$$v_l(k,k') = 2\pi \int_{-1}^{1} d\cos\theta P_l(\cos\theta)\, U\left(\sqrt{k^2 + k'^2 - 2kk'\cos\theta}\right)$$

- evaluated by Gauss-Legendre integration (~ 15 pts)

- transition amplitude and S-matrix

$$\tau_l(E) = -\pi\mu k_0\, t_l(k_0,k_0),$$

$$s_l(E) = 1 + 2i\, \tau_l(E)$$

- phase shifts and inelasticity

$$\delta_l = \frac{1}{2}\arctan\left(\frac{\Re\tau_l}{\frac{1}{2} - \Im\tau_l}\right)$$

$$\eta_l = \sqrt{1 + 4\left[(\Re\tau_l)^2 + (\Im\tau_l)^2 - \Im\tau_l\right]}$$

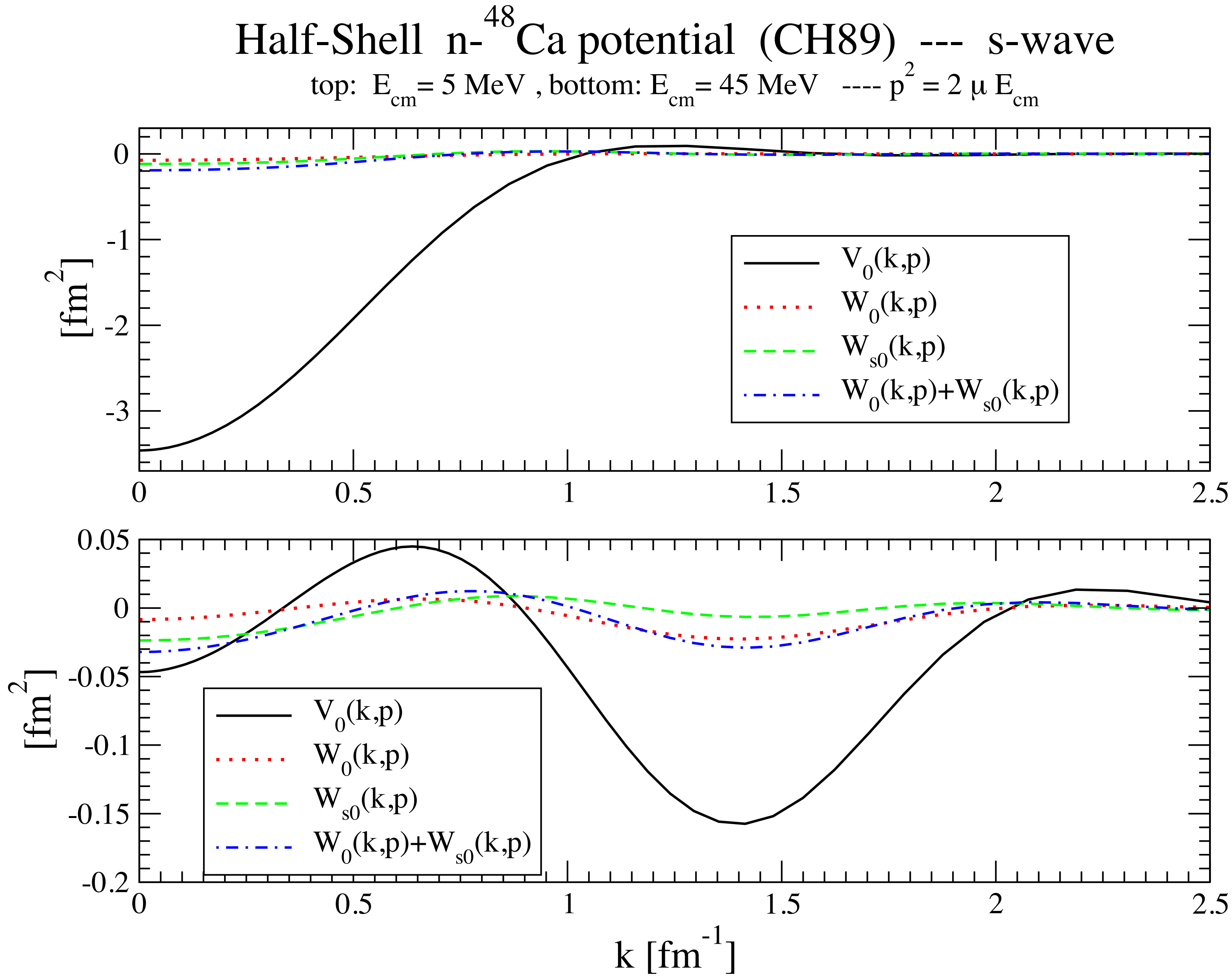


Figure 3: The real and imaginary parts to of the s-wave projected half-shell n+^{48}Ca optical potential as function of k for fixed momentum p, where $p^2 = 2\mu E_{c.m.}$. The top panel shoes $E_{c.m.} = 5$ MeV, the bottom panel $E_{c.m.} = 45$ MeV. The calculations use $R_r = r_r(A^{1/3} + 1)$.

on-shell t-matrix for Wood-Saxon Potential $U_{nucl}(q)$

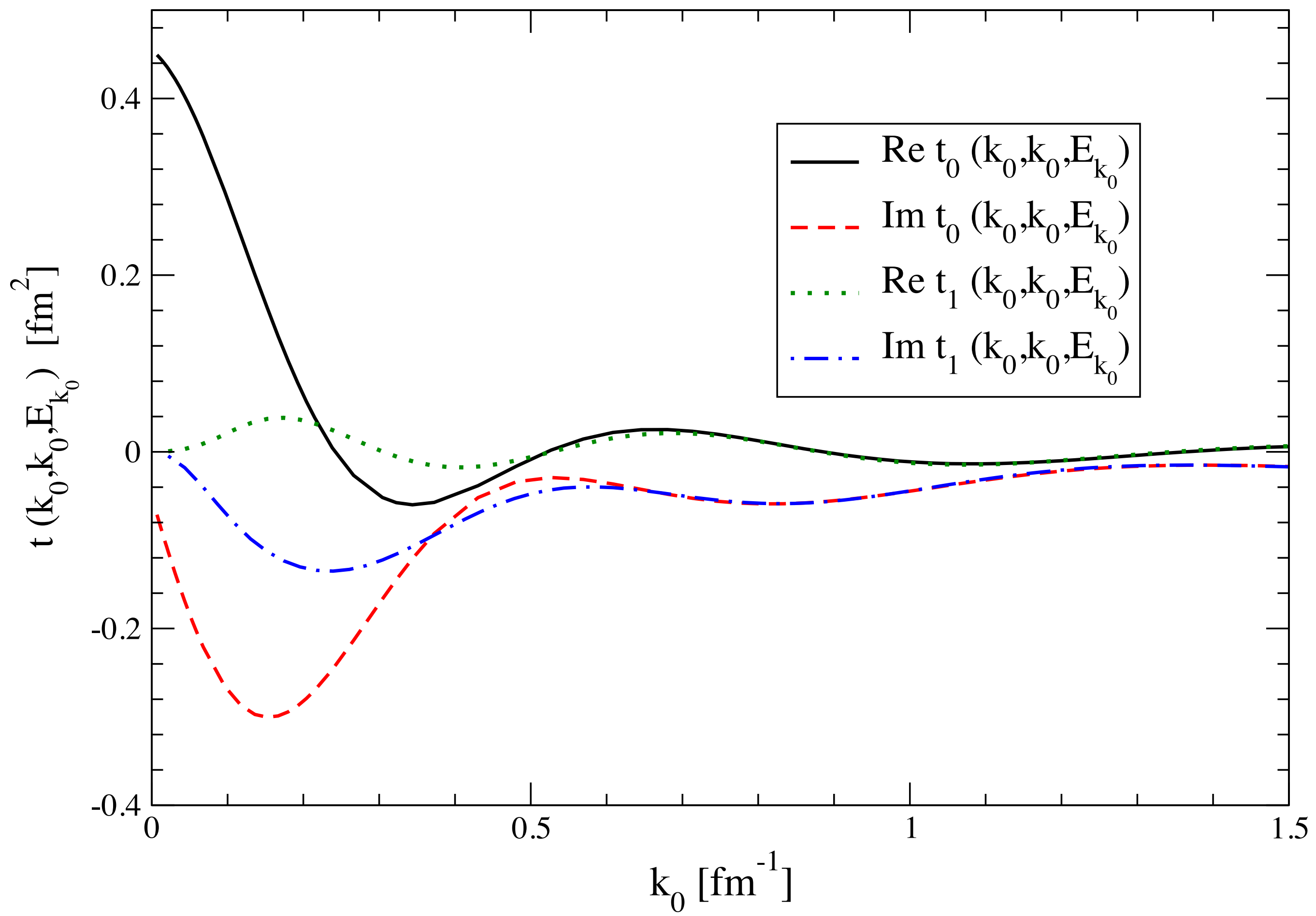


Figure 5: The real and imaginary parts to of the s-wave $(t_0(k_0, k_0, E_{k_0})$ and p-wave $(t_1(k_0, k_0, E_{k_0})$ projected on-shell shell t-matrix solved with the n+^{48}Ca optical potential U_{nucl} as function of the on-shell momentum k_0. The calculations use $R_r = r_r(A^{1/3} + 1)$.

p-space, r-space comparison: phase shifts

E_{lab} [MeV]	E_{cm} [MeV]	δ_0 [deg]		
		k − space	r − space	Neelam
5.00	4.897	-64.068	-64.071	-64.222
10.00	9.794	74.685	74.685	74.570
20.00	19.588	22.463	22.462	22.379
40.00	39.176	-38.026	-38.027	-38.084
50.00	48.970	-58.696	-58.697	-58.746

- k-space : code LH
- r-space : code CE
- Neelam : Fresco

p-space, r-space comparison: bound states

	n = 2	n = 1	n = 2	n = 1
	bound state code		Fresco	
l=0	15.7097	37.805	16.0284	37.9552
l=1	5.1160	28.295	5.4322	28.5426
l=2	17.691		18.03	
l=3	6.5228			

- bound state code : solves the equation

$$\psi_l(p) = \frac{1}{E_{b,l} - \frac{p^2}{2\mu}} \int dp' p'^2 v_l(p,p') \psi_l(p').$$

- returns wave function and binding energy

  solved as linear system with dgeev from lapack

Unitary Pole Approximation (UPA)

- two body t-matrix

$$t = v + vgv$$

- spectral representation

$$\mathbf{1} = \sum_B |\Psi_B\rangle\langle\Psi_B| + \int d^3k |\Psi_k^{(+)}\rangle\langle\Psi_k^{(+)}|$$

- leads to

$$\begin{aligned}\langle\mathbf{k}|t(z)|\mathbf{k}'\rangle &= \langle\mathbf{k}|v|\mathbf{k}'\rangle + \sum_B \frac{\langle\mathbf{k}|v|\Psi_B\rangle\langle\Psi_B|v|\mathbf{k}'\rangle}{z - E_b} \\ &+ \int d^3q \frac{\langle\mathbf{k}|v|\Psi_q^{(+)}\rangle\langle\Psi_q^{(+)}|v|\mathbf{k}'\rangle}{z - E_q} \ .\end{aligned}$$

- Hence for $z \to E_B \equiv -|E_B|$

$$t(z) \simeq \frac{v|\Psi_B\rangle\langle\Psi_B|v}{z + E_B}.$$

UPA

- Ansatz for separable potential

$$V \equiv v|\Psi_B\rangle\langle\Psi_B|v$$

- rank-1 t-matrix

$$t(z) = |h\rangle\tau(z)\langle h| = \frac{|h\rangle\langle h|}{\frac{1}{\lambda} - \langle h|g_0(z)|h\rangle},$$

- at the pole

$$\frac{1}{\lambda} = \langle\Psi_B|vg_0(-E_B)v|\Psi_B\rangle = \langle\Psi_B|v\frac{1}{z+E_B}v|\Psi_B\rangle = \langle\Psi_B|v|\Psi_B\rangle,$$

- t-matrix becomes

$$t(z) = \frac{v|\Psi_B\rangle\langle\Psi_B|v}{\langle\Psi_B|(v - vg_0(z)v)|\Psi_B\rangle}$$

- Explicitly

$$t_l(k', k, z) = \langle k'|v_l|\Psi_B^l\rangle\ \tau_l(z)\ \langle\Psi_B^l|v_l k\rangle = h_B^l(k')\tau_l(z)h_B^l(k)$$

- with

$$h_B^l(k) = \int dk' k'^2\ v_l(k, k')\ \Psi_B^l(k')$$

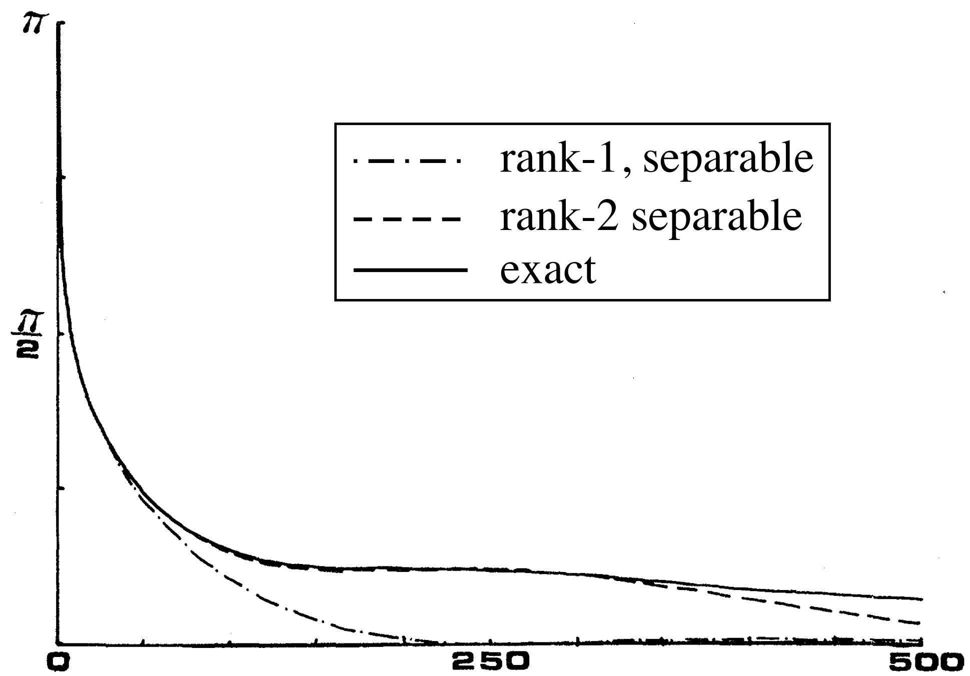
$$\tau_l^{-1}(z) = \int dk k^2 \int dk' k'^2\ \Psi_B^l(k)\ v_l(k, k')\ \Psi_B^l(k') - \int dk k^2 \frac{|h_B^l(k)|^2}{z - \frac{k^2}{2\mu} + i\epsilon}$$

Test case: D.J. Ernst, C.M. Shakin, R. M. Thaler and D. L. Weiss Phys. Rev. C**8,** 2056 (1973)

"deuteron" phase shifts

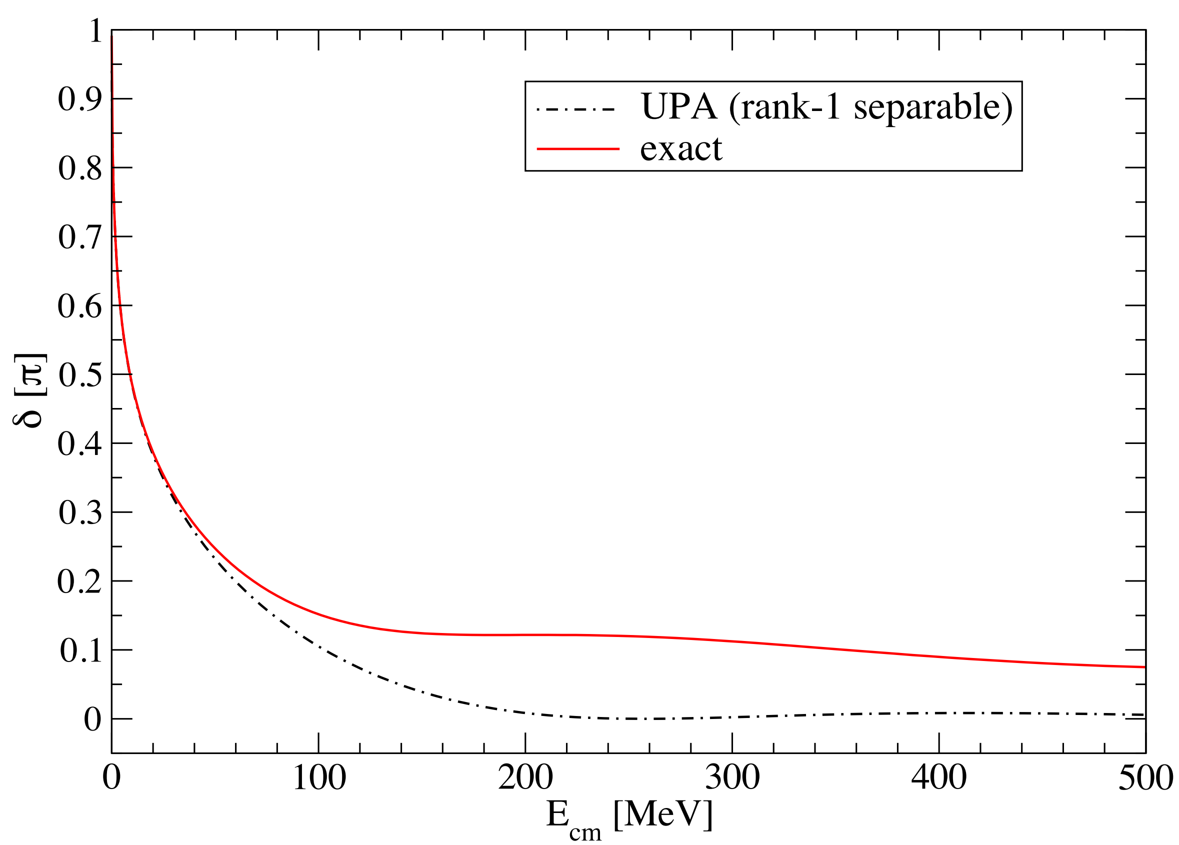
potential: squarewell: V_0=38.5, R=1.93

- ESTW



r-space

- LH



Extension to positive energies (EST)

- Original hamiltonian

$$H = H_0 + V$$

- construct $\mathcal{V} = |v\rangle\lambda\langle v|$ such that at some energy, Ek, the separable hamiltonian is identical to the original one
- for separable potential

$$|\Phi_{k_E}^{(+)}\rangle = |k_E\rangle + \frac{\lambda\langle v|k_E\rangle g_0(E)|v\rangle}{1 - \lambda\langle v|g_0(E_k)|v\rangle}$$

- for the original potential

$$|\Psi_{k_E}^{(+)}\rangle = |k_E\rangle + g_0(E_{k_E})V|\Psi_{k_E}^{(+)}\rangle.$$

- taking $|v\rangle \equiv V|\Psi_{k_E}^{(+)}\rangle$

then $$\frac{1}{\lambda} = \langle\Psi_{k_E}^{(+)}|V|\Psi_{k_E}^{(+)}\rangle,$$

$$\mathcal{V} = \frac{V|\Psi_{k_E}^{(+)}\rangle\langle\Psi_{k_E}^{(+)}|V}{\langle\Psi_{k_E}^{(+)}|V|\Psi_{k_E}^{(+)}\rangle} \equiv |h_k\rangle\lambda\langle h_k|,$$

D. J. Ernst, C. M. Shakin and R. M. Thaler, Phys. Rev. C **8**, 46 (1973).

- t-matrix

$$\langle k'|t(z)|k\rangle \equiv \langle k'|h_k\rangle\tau(z)\langle h_k|k\rangle \equiv \langle k'|h_k\rangle \left[\frac{1}{\lambda} - \langle h_k|g_0(z)|h_k\rangle\right]^{-1} \langle h_k|k\rangle.$$

$$\begin{aligned}\langle p'|t(E)|p\rangle &= \frac{\langle p'|V|\Psi^{(+)}_{k_E}\rangle\langle\Psi^{(+)}_{k_E}|V|p\rangle}{\langle\langle\Psi^{(+)}_{k_E}|V - Vg_0(E)V|\Psi^{(+)}_{k_E}\rangle} \\ t(p',p,E) &= \frac{t^*_{k_E}(p)t_{k_E}(p')}{\langle\Psi^{(+)}_{k_E}|V - Vg_0(E)V|\Psi^{(+)}_{k_E}\rangle},\end{aligned}$$

- explicitly

$$\begin{aligned}&\tau(E)^{-1} = t^*(k_E, k_E, E_{k_E}) \\ &\quad +2\mu\left[\mathcal{P}\int dpp^2\frac{|t(p,k_E,E_{k_E})|^2}{k_E^2 - p^2} - \mathcal{P}\int dpp^2\frac{|t(p,k_E,E_{k_E})|^2}{k_0^2 - p^2}\right] \\ &\quad +i\pi\mu\left[k_0|t(k_0,k_E,E_{k_E})|^2 - k_E|t(k_E,k_E,E_{k_E})|^2\right]\end{aligned}$$

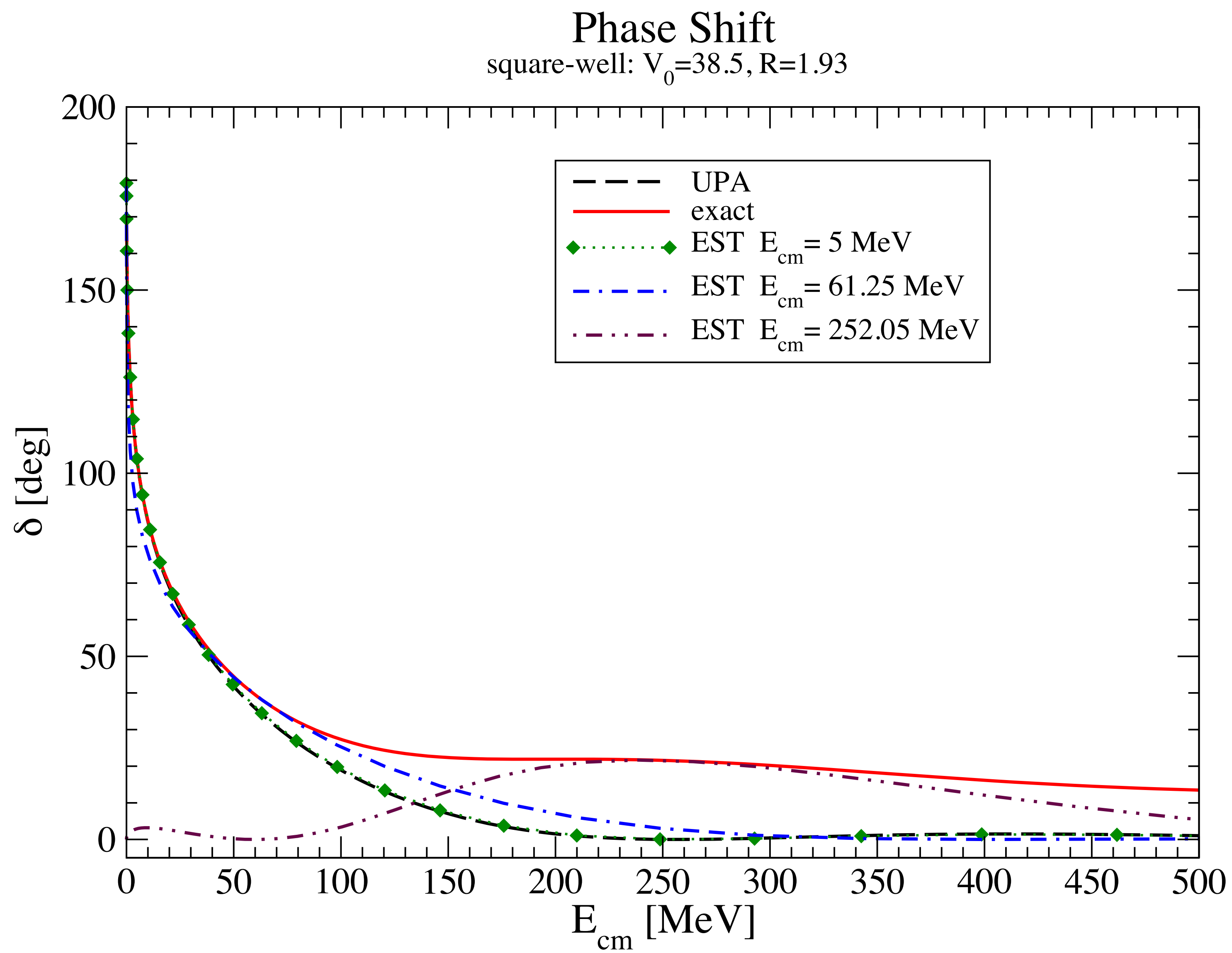
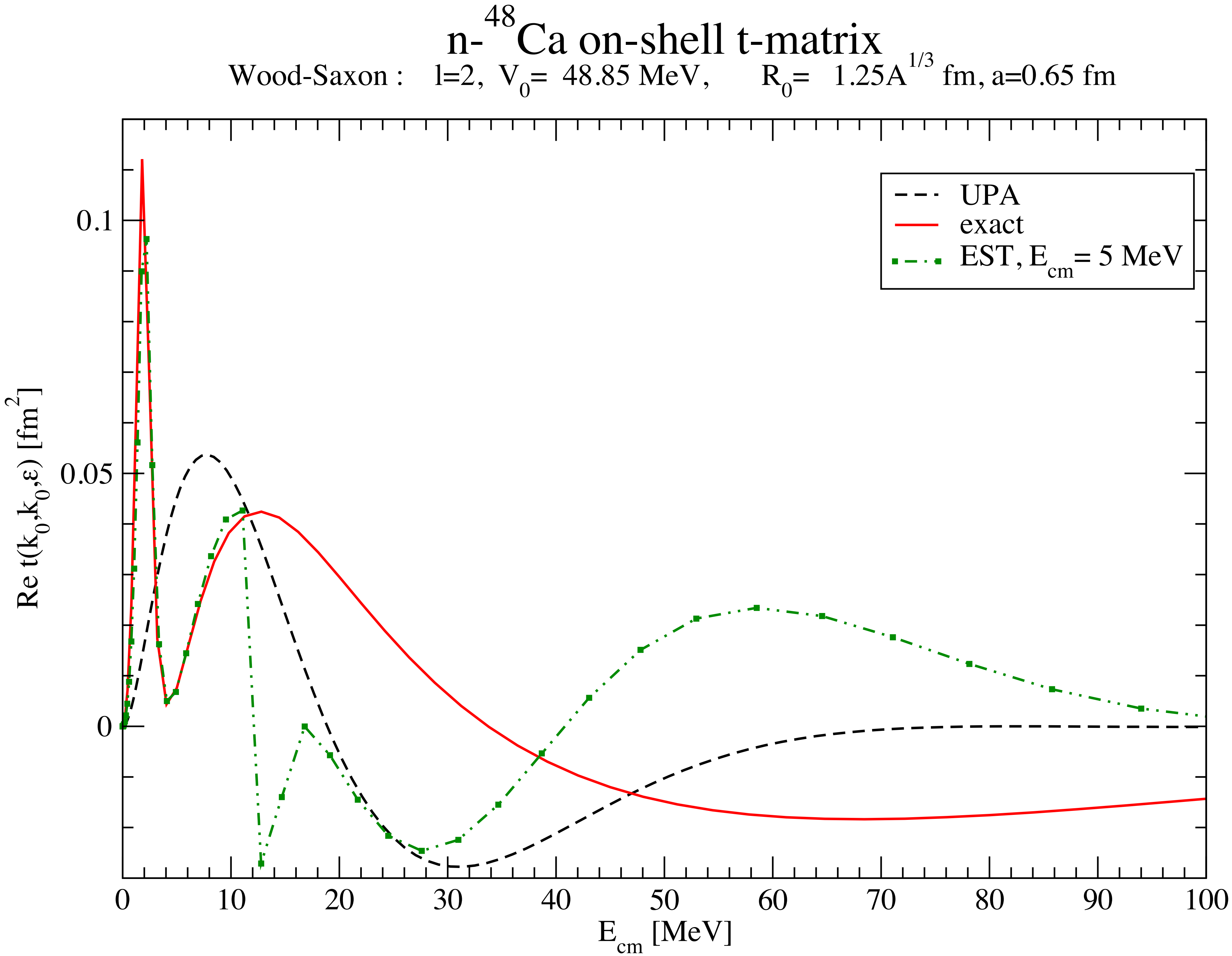


Figure 9: The s-wave for a square well potential with parameters adjusted such that the deuteron binding energy of -2.225 MeV is reproduced. The solid line gives the calculation with the square well potential while the dashed line shows the calculation with the rank-1 UPA approximation. The dotted, double-dash-dotted, and double-dot-dashed lines are calculated with EST rank-1 separable potentials constructed at the energies indicated in the figure.



n-48Ca on-shell t-matrix
Wood-Saxon : l=2, V0= 48.85 MeV, R0= 1.25A1/3 fm, a=0.65 fm
UPA
exact
EST, Ecm= 5 MeV
Re t(k0,k0,ε) [fm2]
Ecm [MeV]
0.1
0.05
0
0
10
20
30
40
50
60
70
80
90
100

Next: Extension to rank-2 potential

- generalized separable potential

$$\mathbf{V} = \sum_{i,j} v|\Psi_i\rangle\langle\Psi_i|M|\Psi_j\rangle\langle\Psi_j|v$$

- constraint

$$\delta_{ik} = \sum_j \langle\Psi_i|M|\Psi_j\rangle\langle\Psi_j|v|\Psi_k\rangle = \sum_j \langle\Psi_i|v|\Psi_j\rangle\langle\Psi_j|M|\Psi_k\rangle.$$

for rank-2

$$\sum_{j=1}^{2} M_{ij}\langle\alpha_j|v|\alpha_k\rangle = \delta_{ik}.$$

- form factor

$$\mathbf{h}(p) = \begin{pmatrix} h_B(p) \\ t(p, k_E, E_{k_E}) \end{pmatrix} = \langle p|V| \begin{pmatrix} |\Psi_B\rangle \\ |\Phi^{(+)}_{k_E}\rangle \end{pmatrix} \equiv \langle p|V| \begin{pmatrix} |\alpha_1\rangle \\ |\alpha_2\rangle \end{pmatrix}.$$

- M-matrix elements

$$\begin{pmatrix} M_{11}M_{12} \\ M_{21}M_{22} \end{pmatrix} \cdot \begin{pmatrix} A_{11}A_{12} \\ A_{21}A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $A_{ij} \equiv \langle\alpha_j|v|\alpha_j\rangle$

D. J. Ernst, C. M. Shakin and R. M. Thaler, Phys. Rev. C **8**, 46 (1973).

- diagonalize M

$$\hat{\mathbf{M}} \equiv \mathbf{U}\mathbf{M}\mathbf{U}^{-1}$$

- eigenvectors

$$|\hat{\alpha}_i\rangle = \sum_j U_{ij}|\alpha_j\rangle$$

- coupling strengths

$$\hat{\lambda}_i = \langle\hat{\alpha}_i|\hat{\mathbf{M}}|\hat{\alpha}_i\rangle$$

- rank-2 separable potential (Neelam and CE)

$$v_l(p',p) = h_{l,1}(p')\lambda_{11}h_{l,1}(p) + h_{l,2}(p')\lambda_{22}h_{l,2}(p)$$

- t-matrix

$$t_l(p',p,E) = \sum_{i,j=1}^{2} h_{l,i}(p')\tau_{ij}(E)h_{l,j}(p)$$

with

$$\tau_{ij}(E) = (\lambda - B(E))^{-1}_{ij} \,,$$

where

$$B_{ij}(E) \equiv \int dp''p''^2 \frac{h_i(p'')h_j(p'')}{E - p^2/2\mu}$$

OR Calculate t-matrix without diagonalizing M

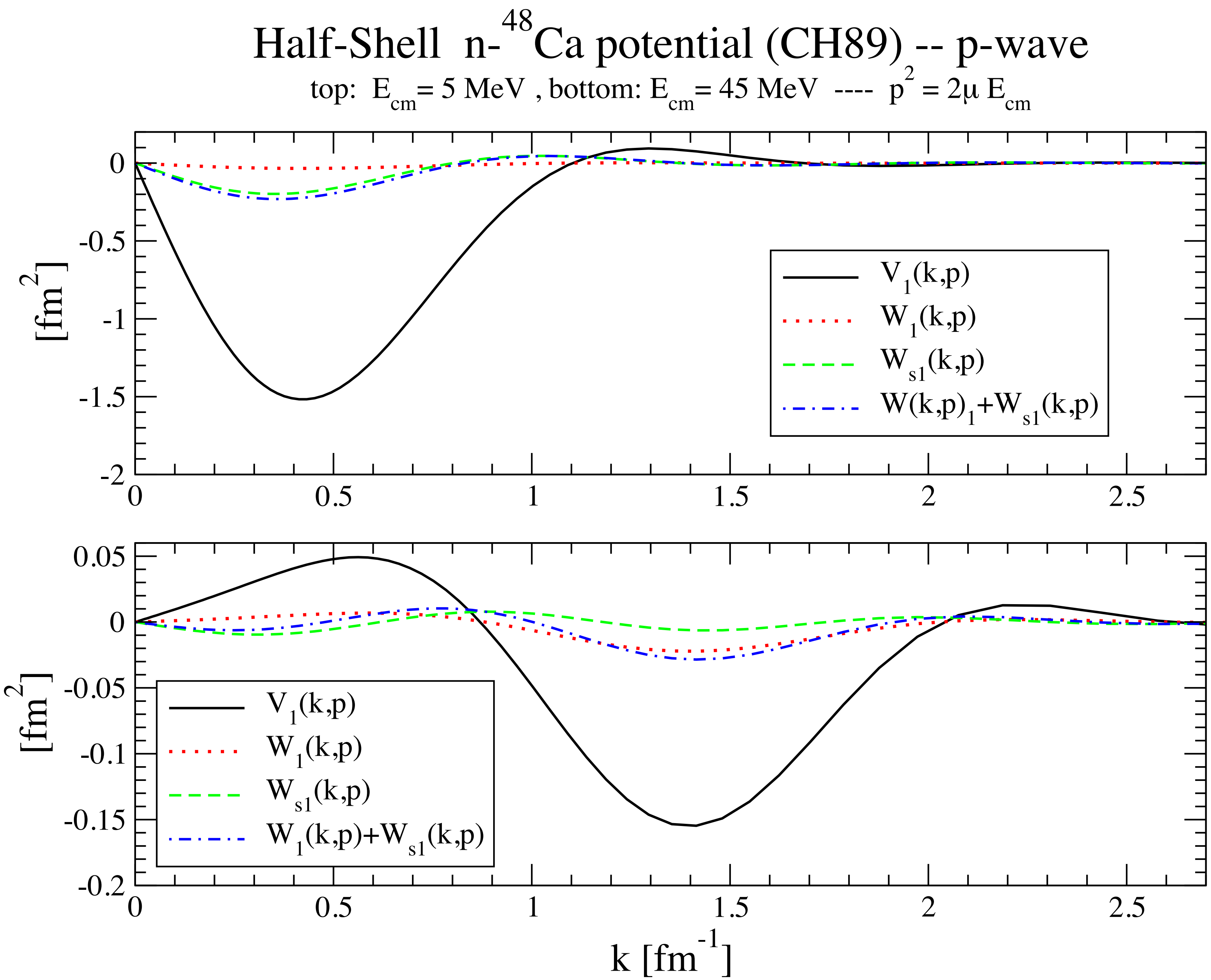


Figure 4: The real and imaginary parts to of the p-wave projected half-shell n+^{48}Ca optical potential as function of k for fixed momentum p, where $p^2 = 2\mu E_{c.m.}$. The top panel shoes $E_{c.m.} = 5$ MeV, the bottom panel $E_{c.m.} = 45$ MeV. The calculations use $R_r = r_r(A^{1/3} + 1)$.

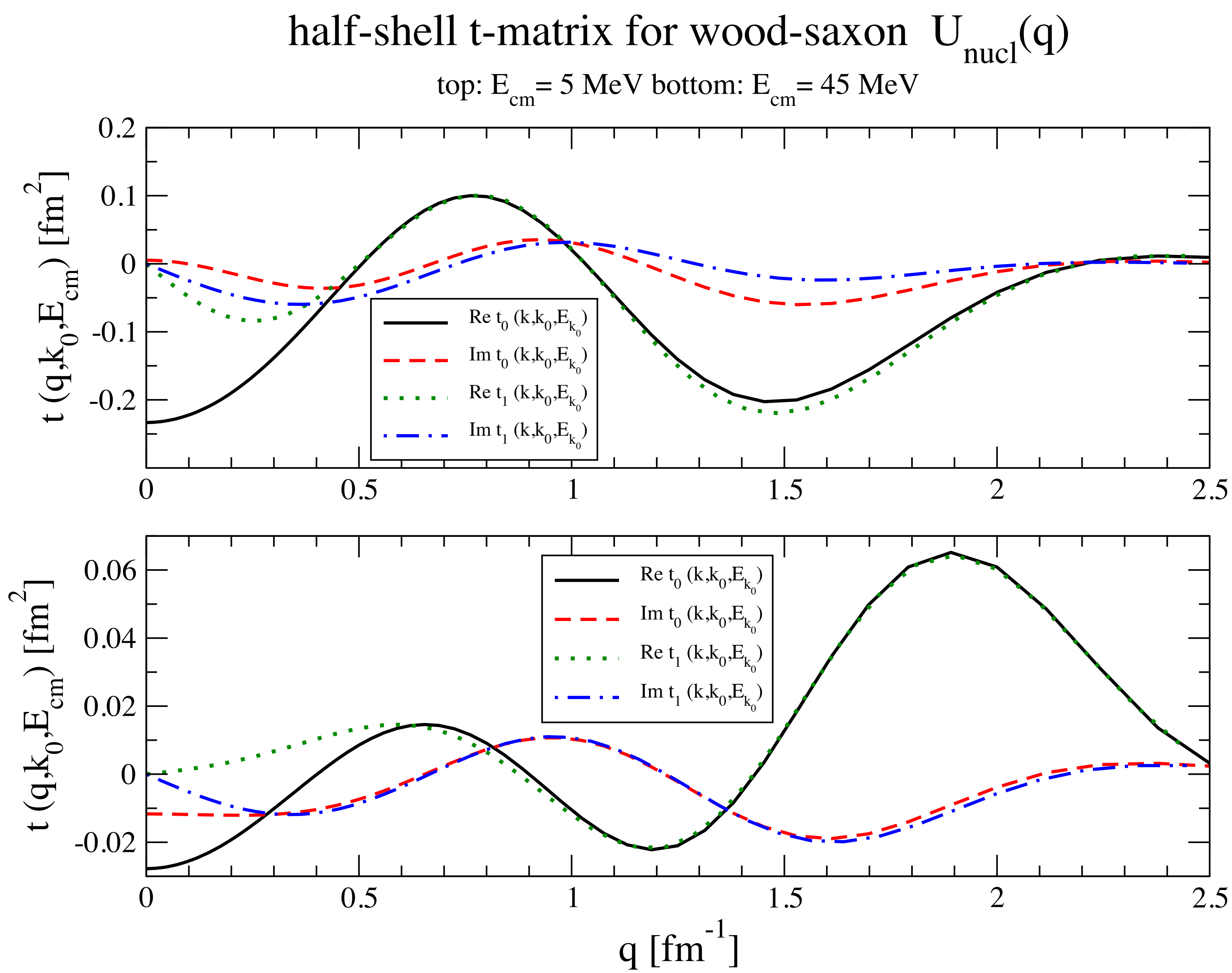
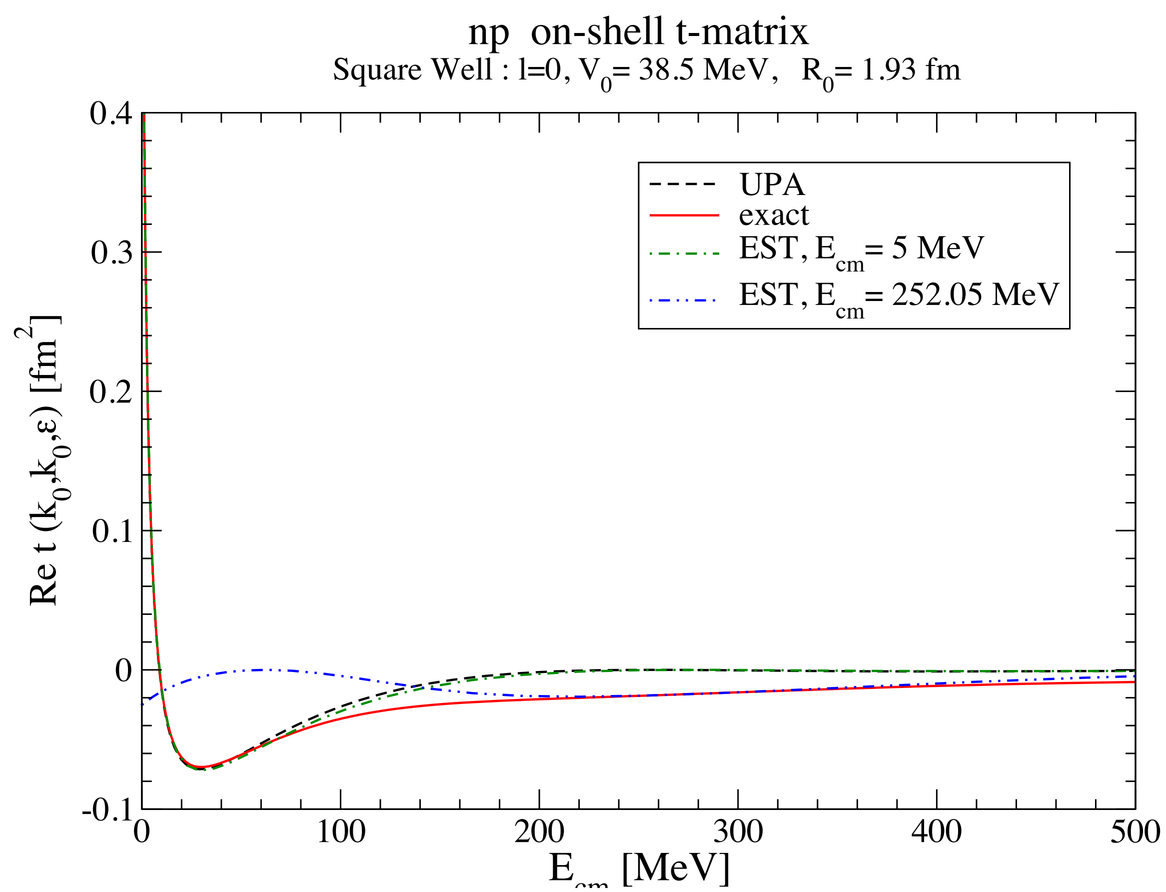
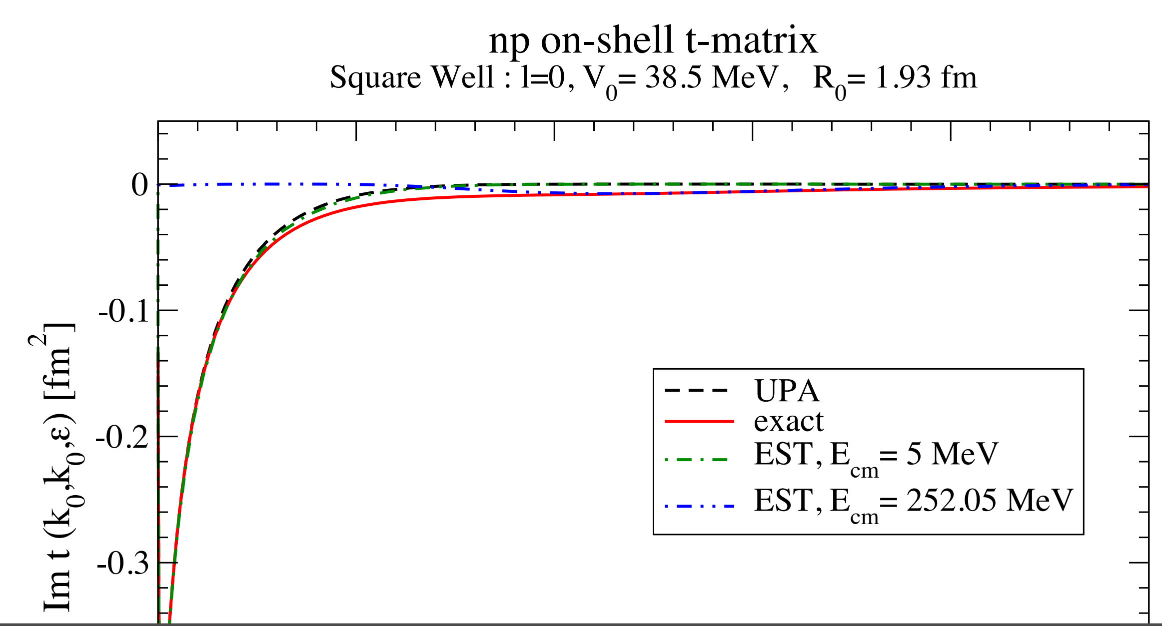


Figure 6: The real and imaginary parts to of the s-wave ($t_0(q, k_0, E_{k_0})$ and p-wave ($t_1(q, k_0, E_{k_0})$ projected half-shell t-matrix solved with the n+^{48}Ca optical potential U_{nucl} as function of q for fixed momentum k_0, where $k_0^2 = 2\mu E_{c.m.}$. The top panel shoes $E_{c.m.} = 5$ MeV, the bottom panel $E_{c.m.} = 45$ MeV. The calculations use $R_r = r_r(A^{1/3} + 1)$.


np on-shell t-matrix
Square Well : l=0, V0= 38.5 MeV, R0= 1.93 fm
Re t (k0,k0,ε) [fm2]
0.4
0.3
0.2
0.1
0
-0.1
0
100
200
300
400
500
Ecm [MeV]
UPA
exact
EST, Ecm= 5 MeV
EST, Ecm= 252.05 MeV


np on-shell t-matrix
Square Well : l=0, V0= 38.5 MeV, R0= 1.93 fm
Im t (k0,k0,ε) [fm2]
0
-0.1
-0.2
-0.3
UPA
exact
EST, Ecm= 5 MeV
EST, Ecm= 252.05 MeV